Matrices: Gaussian & Gauss-Jordan Elimination

Definition: A **system of equations** is a collection of two or more equations with the same set of unknown variables that are considered simultaneously.

Ex: The following set of equations is a system of equations.

$$x - 2y + 3z = 9$$
$$-x + 3y \quad = -4$$
$$2x - 5y + 5z = 17$$

Definition: An **augmented matrix** is a rectangular array of numbers that represents a system of equations.

Ex: Turn the following system of equations into an augmented matrix.

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y \quad = -4 \\ 2x - 5y + 5z = 17 \end{cases} \qquad \text{Becomes: } \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

Gaussian elimination

Gaussian elimination is a method for solving systems of equations in matrix form.

Goal: turn matrix into **row-echelon form** $\left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right]$.

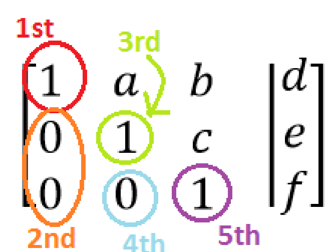
Once in this form, we can say that $z = f$ and use back substitution to solve for y and x.

3 Elementary Row Operations:

1) Exchange two rows.
   (Written $R_i \leftrightarrow R_j$)
2) Multiply a row by a non-zero constant.
   (Written $\#R_i \rightarrow R_i$ or $\#R_i \rightarrow NR_j$)
3) Add a multiple of a row to another row.
   (Written $\#R_i + R_j \rightarrow R_j$ or $\#R_i + R_j \rightarrow NR_j$)

Use the elementary row operations and follow these steps:

1) Get a 1 in the first column, first row
2) Use the 1 to get 0's in the remainder of the first column
3) Get a 1 in the second column, second row
4) Use the 1 to get 0's in the remainder of the second column
5) Get a 1 in the third column, third row



Note: It is not *necessary* to solve the matrix in this order; however, this approach is often the most direct.

Ex: Solve the following set of equations:

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \quad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \quad R_1 + R_2 \to R_2 \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{array}\right] \quad -2R_1 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & 1 & -1 \end{array}\right] \quad R_2 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 4 & 4 \end{array}\right] \quad \frac{1}{4} R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

We are left with the three new equations:

$$\begin{gathered} x + \ -2y + 3z = 9 \\ y + 3z = 5 \\ z = 1 \end{gathered}$$

Based on the last variable we can use back substitution to find the remaining values.

Solutions are $x = 10, y = 2, and\ z = 1$.

Gauss-Jordan elimination

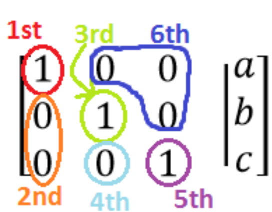
Gauss-Jordan elimination is another method for solving systems of equations in matrix form. It is really a continuation of Gaussian elimination.

Goal: turn matrix into **reduced row-echelon form** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$.

Once it is in this form, we can say $x = a, y = b, and\ z = c$ or $(x, y, z) = (a, b, c)$.

Use same row operations as before.

The steps are slightly different because we need zeros above the diagonal line of 1's as well as below. We can either complete Gaussian elimination and then work on the 0's above the 1's, **or** work on the zeros above as we move through the rows, as demonstrated below.



or

Once the values are found we can always check by plugging back into original equation.