Matrices: Determinants

Determinant: A real number related to an $n \times n$ square matrix A, denoted as |A|.

For a **2×2 matrix**, the determinant is found by multiplying entries (elements) diagonally and subtracting.

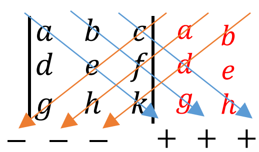
The determinant of this matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.

For a **3×3 matrix**, the determent can be found two ways. The first is the **method of diagonals**.

Steps: 1) Copy and paste the first and second columns of the matrix onto the right side of the matrix.

2) Multiply along the diagonals.

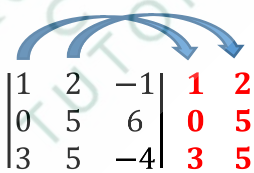
3) Add and subtract as shown below.



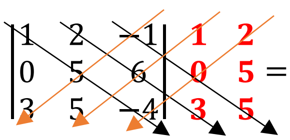
$= aek + bfg + cdh - ceg - afh - bdk$ or $(aek + bfg + cdh) - (ceg + afh + bdk)$

Ex. Find the determinant of $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \\ 3 & 5 & -4 \end{bmatrix}$.

Step 1) Copy and paste the first two columns of the matrix onto the right side of the matrix



Step 2) Multiply along the diagonals in each direction. The product of elements from top left to bottom right are grouped together and the product of elements from the top right to bottom left are grouped together. Then we find their difference.



$= (1)(5)(-4) + (2)(6)(3) + (-1)(0)(5) - (-1)(5)(3) - (1)(6)(5) - (2)(0)(-4)$

$= -20 + 36 + 0 + 15 - 30 + 0$
$= 1$

We can also find the determinant using **cofactor expansion** (this works for any size matrix).

Steps: 1) Pick one row or column.

2) Multiply each term in that row or column by its corresponding **cofactor**.

3) Add the results.

The **minor** of the entry a_{ij}, denoted as M_{ij}, is the determinant of a square matrix found by deleting the i[th] row and the j[th] column from some larger square matrix.

Ex: The minor M_{12} of the of the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ is $\begin{vmatrix} d & f \\ g & k \end{vmatrix} = dk - gf.$

The **cofactor** of the entry a_{ij} is its minor M_{ij} multiplied by $(-1)^{i+j}$, written $A_{ij} = (-1)^{i+j}M_{ij}$.

Ex: The cofactor A_{12} of the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ is $(-1)^{1+2}\begin{vmatrix} d & f \\ g & k \end{vmatrix} = (-1)^3(dk - gf)$

$$= -dk + gf.$$

Note: Another way to find the cofactor of an element is to multiply its minor by the sign of its corresponding location in the **array of signs**: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Ex. Find the determinant of $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \\ 3 & 5 & -4 \end{bmatrix}$.

Step 1) Pick a column. We'll use column 1: $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

Steps 2) and 3) Multiply each term in column 1 by corresponding cofactor and add.

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \\ 3 & 5 & -4 \end{vmatrix} = (\mathbf{1})\begin{vmatrix} 5 & 6 \\ 5 & -4 \end{vmatrix} - (\mathbf{0})\begin{vmatrix} 2 & -1 \\ 5 & -4 \end{vmatrix} + (\mathbf{3})\begin{vmatrix} 2 & -1 \\ 5 & 6 \end{vmatrix} = \mathbf{1}(-20 - 30) - \mathbf{0}(-8 + 5) + \mathbf{3}(12 + 5) = 1$$