CRAFTON HILLS COLLEGE TUTORING CENTER

Matrices: Cramer's Rule

Cramer's Rule is a method of solving systems of equations using determinants.

The following is **Cramer's Rule with two variables:**

Consider the system of equations $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$.

Let $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_x = \begin{vmatrix} \boldsymbol{c_1} & b_1 \\ \boldsymbol{c_2} & b_2 \end{vmatrix}$, $D_y = \begin{vmatrix} a_1 & \boldsymbol{c_1} \\ a_2 & \boldsymbol{c_2} \end{vmatrix}$

- D: determinant of the coefficient matrix
- D_x: determinant of the matrix formed by **replacing the x-column with the constants**
- D_y: determinant of the matrix formed by **replacing the y-column with the constants**

If $D \neq 0$, then $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$; the solution to the system of equations is the ordered pair $\left(\frac{D_x}{D}, \frac{D_y}{D}\right)$.

(Note: If $D = 0$, Cramer's Rule does not apply—use a different method to solve the system of equations).

(Ex.) Solve the system of equations using Cramer's Rule, if applicable. $\begin{cases} 2x + 5y = 11 \\ -3x + y = -4 \end{cases}$

Find the value of each determinant $D,\ D_x,\ and\ D_y$.

$$D = \begin{vmatrix} 2 & 5 \\ -3 & 1 \end{vmatrix} = (2)(1) - (5)(-3) = 2 + 15 = 17$$

$$D_x = \begin{vmatrix} 11 & 5 \\ -4 & 1 \end{vmatrix} = (11)(1) - (5)(-4) = 11 + 20 = 31$$

$$D_y = \begin{vmatrix} 2 & 11 \\ -3 & -4 \end{vmatrix} = (2)(-4) - (11)(-3) = -8 + 33 = 25$$

Thus, $x = \frac{D_x}{D} = \frac{31}{17}$ and $y = \frac{D_y}{D} = \frac{25}{17}$. The solution to the system of equations is the ordered pair $\left(\frac{31}{17}, \frac{25}{17}\right)$.

Cramer's Rule can also be extended to solve systems of linear equations in three variables:

Consider the system of equations $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_x = \begin{vmatrix} \boldsymbol{d_1} & b_1 & c_1 \\ \boldsymbol{d_2} & b_2 & c_2 \\ \boldsymbol{d_3} & b_3 & c_3 \end{vmatrix}$, $D_y = \begin{vmatrix} a_1 & \boldsymbol{d_1} & c_1 \\ a_2 & \boldsymbol{d_2} & c_2 \\ a_3 & \boldsymbol{d_3} & c_3 \end{vmatrix}$, $D_z = \begin{vmatrix} a_1 & b_1 & \boldsymbol{d_1} \\ a_2 & b_2 & \boldsymbol{d_2} \\ a_3 & b_3 & \boldsymbol{d_3} \end{vmatrix}$

D	D_x	D_y	D_z
determinant of the coefficient matrix	determinant of the matrix formed by **replacing the x- column with the constants**	determinant of the matrix formed by **replacing the y-column with the constants**	determinant of the matrix formed by **replacing the z-column with the constants**

If $D \neq 0$, then $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$; the solution to the system of equations is the ordered triple $\left(\frac{D_x}{D}, \frac{D_y}{D}, \frac{D_z}{D}\right)$.

(Note: If $D = 0$, Cramer's Rule does not apply—use a different method to solve the system of equations).

Ex. Solve the system of equations using Cramer's Rule, if applicable. $\begin{cases} 2x + 3y - z = -12 \\ x - y - z = -4 \\ -4x + 3y + z = 14 \end{cases}$

Find the value of each determinant $D,\ D_x,\ D_y\ and\ D_z$.

$$D = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -1 \\ -4 & 3 & 1 \end{vmatrix} = 2\begin{vmatrix} -1 & -1 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix}$$
$$= 2(-1+3) - 1(3+3) - 4(-3-1)$$
$$= 2(2) - 1(6) - 4(-4)$$
$$= 14$$

$$D_x = \begin{vmatrix} -12 & 3 & -1 \\ -4 & -1 & -1 \\ 14 & 3 & 1 \end{vmatrix} = -12\begin{vmatrix} -1 & -1 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & -1 \\ 3 & 1 \end{vmatrix} + 14\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix}$$
$$= -12(-1+3) + 4(3+3) + 14(-3-1)$$
$$= -12(2) + 4(6) + 14(-4)$$
$$= -56$$

$$D_y = \begin{vmatrix} 2 & -12 & -1 \\ 1 & -4 & -1 \\ -4 & 14 & 1 \end{vmatrix} = 2\begin{vmatrix} -4 & -1 \\ 14 & 1 \end{vmatrix} - 1\begin{vmatrix} -12 & -1 \\ 14 & 1 \end{vmatrix} - 4\begin{vmatrix} -12 & -1 \\ -4 & -1 \end{vmatrix}$$
$$= 2(-4+14) - 1(-12+14) - 4(12-4)$$
$$= 2(10) - 1(2) - 4(8)$$
$$= -14$$

$$D_z = \begin{vmatrix} 2 & 3 & -12 \\ 1 & -1 & -4 \\ -4 & 3 & 14 \end{vmatrix} = 2\begin{vmatrix} -1 & -4 \\ 3 & 14 \end{vmatrix} - 1\begin{vmatrix} 3 & -12 \\ 3 & 14 \end{vmatrix} - 4\begin{vmatrix} 3 & -12 \\ -1 & -4 \end{vmatrix}$$
$$= 2(-14+12) - 1(42+36) - 4(-12-12)$$
$$= 2(-2) - 1(78) - 4(-24)$$
$$= 14$$

Thus, $x = \frac{D_x}{D} = \frac{-56}{14} = -4$, $y = \frac{D_y}{D} = \frac{-14}{14}$=-1, $and\ z = \frac{D_y}{D} = \frac{14}{14}$=1.

The solution to the system of equations is the ordered triple $(-4, -1, 1)$.